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LETTER TO THE EDITOR

Critical exponents for the four-state Potts model

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Abstract. We obtain a number of relations connecting the critical exponents of the Ashkin–Teller model. For the special case of the four-state Potts model these relations predict $\alpha = \frac{2}{3}$, $\beta = \frac{1}{12}$, $\gamma = \frac{7}{6}$, $\delta = 15$. Series estimates are in agreement with these predictions.

Although it is over thirty years since the Ashkin–Teller model was first described (Ashkin and Teller 1943) it is only recently that very much has been learnt of its properties. The model can be regarded as consisting of a regular lattice with two Ising spins σ_i, $S_i = \pm 1$ on each site, and having the Hamiltonian

$$H = -\sum_{\text{bonds}} (J\sigma_i\sigma_j + J'S_iS_j + J_4\sigma_i\sigma_jS_iS_j). \tag{1}$$

In the general case Wu and Lin (1974) have shown that two phase transitions are expected, but for the present we consider only $J' = J, J_4 \leqslant J$ so that only one transition is expected.

Since we have two Ising subsystems coupled by a four-spin interaction the model is similar to the eight-vertex model and so, as explained by Kadanoff and Wegner (1971), the critical exponents should vary continuously with J_4/J. A special case of the Ashkin–Teller model is $J = J' = J_4$, which is equivalent to the four-state Potts model (Potts 1952). It has been shown (Baxter 1973) that the transition is continuous.

In the eight-vertex and Ashkin–Teller models there are two natural order parameters, generally called the magnetization M and the polarization P.

For the Ashkin–Teller model

$$P = \langle \sigma_i S_i \rangle \tag{2}$$

$$M = \langle \sigma_i \rangle = \langle S_i \rangle. \tag{3}$$

The critical behaviour of P and M is described by the exponents β_e, β_m respectively, and we can conjecture two sets of critical exponents with the usual scaling laws applying to each set. For instance,

$$\alpha' + \beta_e(1+\delta_e) = 2 \tag{4}$$

$$\alpha' + \beta_m(1+\delta_m) = 2. \tag{5}$$

In the eight-vertex model (4) has been confirmed by the work of Baxter and Kelland (1974) and (5) by the work of Gaunt (1974). In addition to these scaling relations we conjecture that the following two relations apply to the Ashkin–Teller models:

$$\delta_{\mathrm{m}} = 15 \tag{6}$$

$$\beta_{\mathrm{e}} = \tfrac{1}{4} - \tfrac{1}{4}\alpha'. \tag{7}$$

While there have been discussions of the significance of fixed δ_{m} in three dimensions (Gunton and Buckingham 1968, Brout 1971), the significance of $\delta_{\mathrm{m}} = 15$ in two dimensions remains rather poorly understood, but the relation has been confirmed by series analysis for $J_4/J = 0$ and 1. Relation (7) appears to be a similar type of relation, lying outside scaling and yet it applies to the eight-vertex model (Baxter 1974) and to the triangular lattice triplet model with pure three-spin interactions (Baxter *et al* 1975). Perturbation calculations show that both (6) and (7) should apply to the Ashkin–Teller model on all two-dimensional lattices, at least to first order in J_4/J.

For the Ashkin–Teller model with $J_4 = J = J'$ we make the transformation

$$\sigma_i \rightarrow \sigma_i S_i \qquad S_i \rightarrow S_i$$

which leaves (1) unchanged and interchanges M and P. We thus have

$$\beta_{\mathrm{m}} = \beta_{\mathrm{e}} \qquad (J_4 = J). \tag{8}$$

Solving (5), (6), (7) and (8) gives

$$\alpha' = \tfrac{2}{3} \tag{9}$$

$$\beta_{\mathrm{e}} = \beta_{\mathrm{m}} = \tfrac{1}{12}. \tag{10}$$

Assuming the scaling laws gives

$$\gamma_{\mathrm{m}} = \gamma'_{\mathrm{m}} = \gamma_{\mathrm{e}} = \gamma'_{\mathrm{e}} = \tfrac{7}{6} \tag{11}$$

$$\delta_{\mathrm{e}} = \delta_{\mathrm{m}} = 15. \tag{12}$$

These are the same exponents as for the triplet model.

It should be noted that these predictions are independent of which two-dimensional lattice is considered. This is somewhat surprising as in general if $J' = J$ the manner in which the exponents depend on J_4 is lattice-dependent. In particular

$$\left.\frac{\partial}{\partial(J_4/J)}\beta_{\mathrm{m}}\right|_{J_4=0} = -\tfrac{1}{4}Ak_{\mathrm{B}}T_{\mathrm{c}}/qJ \tag{13}$$

where q is the coordination number of the lattice and A is the specific heat amplitude (at $J_4 = 0$). It appears that only at points of special symmetry, $J_4/J = 0$ or 1, are the exponents lattice-independent.

The predictions (9) to (12) disagree with the conjecture of Ditzian (1972) that the eight-vertex and square lattice Ashkin–Teller models should have the same exponents for the same values of J_4/J.

The predictions (9) and (10) have been tested by analysis of low-temperature series. The notation used follows Enting (1974a). Expansions for the zero-field partition function are known through to u^{16} for the square lattice (Kihara *et al* 1954). The full field dependence of square lattice terms through to u^{13} is given by Straley and Fisher

(1973). Series expansions for the triangular lattice have been calculated using the configurational data of Sykes *et al* (1965).

$$\begin{aligned}\ln \Lambda = {} & 3\mu u^6+9\mu^2u^{10}+18\mu^2u^{11}+(6\mu^3-31\tfrac{1}{2}\mu^2)u^{12}+(9\mu^4+63\mu^3)u^{14}+120\mu^3u^{15}\\ & +(18\mu^5+72\mu^4-162\mu^3)u^{16}+(144\mu^4-540\mu^3)u^{17}\\ & +(3\mu^7+42\mu^6+135\mu^5+195\mu^4+522\mu^3)u^{18}+(252\mu^5+1044\mu^4)u^{19}\\ & +(90\mu^7+306\mu^6-414\mu^5-1714\tfrac{1}{2}\mu^4)u^{20}\\ & +(36\mu^7+756\mu^6+2196\mu^5-6162\mu^4)u^{21}+\dots \qquad \text{(triangular)}. \end{aligned} \tag{14}$$

The conventional form of the order parameter for the Potts model does not correspond directly to either P or M but since $\beta_{\mathrm{e}} = \beta_{\mathrm{m}}$ any choice of order parameter should lead to the same exponent.

On constructing Padé approximants to the order and to the specific heat, the estimates obtained were:

$$\alpha' = 0{\cdot}64 \pm 0{\cdot}05 \qquad \text{(sq)} \tag{15}$$

$$\beta = 0{\cdot}089 \pm 0{\cdot}03 \qquad \text{(sq)} \tag{16}$$

$$\alpha' = 0{\cdot}6 \pm 0{\cdot}1 \qquad \text{(tri)} \tag{17}$$

$$\beta = 0{\cdot}089 \pm 0{\cdot}03 \qquad \text{(tri)}. \tag{18}$$

The ranges given denote the spread of the estimates. It seems that the observed spread is not due to non-physical singularities since transforming the series did not reduce the spread of estimates. If any confluent singularities are present (and this is quite a reasonable explanation for the spread of estimates) then the ranges given in (15) to (18) may well be rather less than the errors in the exponent estimates. This would explain the discrepancy in the β estimates.

On the whole, the series confirm (9) and (10) as well as can be expected from such short series. On the square lattice $\delta = 15$ has been tested for the Potts model (Enting 1974b), but on the triangular lattice there are not enough high-field polynomials known for useful estimates to be obtained.

Of rather more interest than the actual values of the exponents is the manner in which the predictions (9) to (12) were obtained. It is very rarely that the critical exponents can be obtained directly from general exponent relations. The relation (7) is essentially in the form given by Enting and Gaunt (1974). There is, however, no indication of what relation of this type would be most fundamental, fundamental that is in the sense that it may be possible to obtain a proof of the relation or of a corresponding inequality. Possible alternative forms are $\gamma_{\mathrm{e}} = \nu+\frac{1}{2}$ or $\delta_{\mathrm{e}} = (\frac{7}{2}-\frac{3}{2}\alpha)/(\frac{1}{2}-\frac{1}{2}\alpha)$. A more complicated possibility is that some of the constants in these relations need to be expressed in terms of the constants $\delta_{\mathrm{m}} = 15$ or $\eta_{\mathrm{m}} = \frac{1}{4}$ before the true physical significance becomes apparent.

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